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Dynamics of a non Axiom A polynomial skew product

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In this note, the dynamics of a non Axiom A polynomial skew product on $\mathbb{C}^2$ is investigated. We will show that the fiber Julia set J_x is a circle for any $0 < x < 1$ and that J_x tends to the fiber filled Julia set K_1 as $x \to 1$.

1 Introduction

In this note, we consider a regular polynomial skew product on $\mathbb{C}^2$ of the form :

$$f(z, w) = (p(z), q_z(w)) = (z^2, w^2 + 2(1 - z)w).$$

It was first studied in Jonsson [J]. He has shown that it has a saddle fixed point $(1, 0)$ which lies in the second Julia set J_2. Consequently, it is not Axiom A. His argument also says that $J_p \times \{0\} \subset J_2$. Thus all saddle periodic points in $J_p \times \{0\}$ lie in J_2.

Consider the point $(x, 0)$ for $0 < x < 1$. Its forward orbit satisfies

$$f^n(x, 0) = (x^{2^n}, 0) \to (0, 0) \quad (n \to +\infty),$$

while one of its backward orbit on the real plane satisfies

$$f^{-n}(x, 0) \ni (x^{1/2^n}, 0) \to (1, 0) \quad (n \to +\infty).$$

Thus these points $(x, 0)$ belong to $W^s(\alpha) \cap W^u(\hat{\beta})$, where $\alpha = (0, 0)$ and $\beta = (1, 0)$ are saddle fixed points of f and $\hat{\beta} = (\cdots, \beta, \beta)$ is the fixed prehistory

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of β. Then, from the general theory, it follows that the fiber Julia set J_z does not depend continuously on z. We will show that $J_x \to K_1 \neq J_1$ as $x \to 1$.

2 Dynamics on the real plane

In this and the next sections, we consider the dynamics of f on $\mathbb{R}^2$. Let K^{re} be the set of points of f in $\mathbb{R}^2$, whose orbits are bounded :

$$K^{re} = \{(x, y) \in \mathbb{R}^2; \{f^n(x, y)\}_{n \geq 0} \text{ is bounded}\}.$$

We denote its vertical slice by $K^{re}_x = \{y \in \mathbb{R}; (x, y) \in K^{re}\}$. Then $K^{re}_1 = \{-1 \leq y \leq 1\}$ and $K^{re}_{-1} = q^{-1}_{-1}(K^{re}_1)$. Put

$$K' = \{-1 \leq x \leq 1,\ 2(x-1) \leq y \leq 0,\ y^2 + 2(1-x)y \geq 2(x^2 - 1)\}.$$

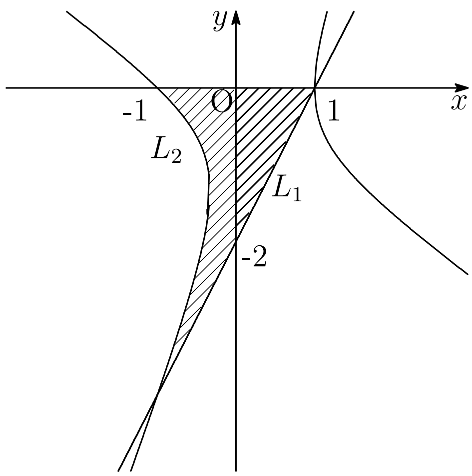

Figure 1: The set K'

The shaded region in Figure 1 indicates the set K'. Here is the main result of this section.

Theorem 2.1. $K^{re} = K' \cup (\{1\} \times K^{re}_1) \cup (\{-1\} \times K^{re}_{-1})$.

It is evident that $K^{re} \subset \{-1 \leq x \leq 1\}$. Put $U_0 = \{-1 < x < 1,\ y > 0\}$. It follows that $f(U_0) \subset U_0$.

Lemma 2.1. *$U_0 \subset \mathbb{R}^2 \setminus K^{re}$.*

proof. Take a point $(x, y) \in U_0$ and put $(x_n, y_n) = f^n(x, y)$. We may assume that $x \geq 0$ since $x_1 > 0$ for $x < 0$. Then $q_x(y) > y$ if and only if $y > 2x - 1$. Since $0 \leq x < 1$ and $y_n > 0$ for any n, there exists $k \geq 0$ such that $y_k > 2x_k - 1$. Then $y_{k+1} > y_k > 2x_k - 1 > 2x_{k+1} - 1$. We can repeat this argument and we conclude that the sequence $\{y_n\}_{n \geq k}$ is monotonely increasing. If it is bounded, it converges to $y_0 \in \mathbb{R}$. Taking limit of the sequence $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$, we have $(0, y_0) = (0, q_0(y_0)) = (0, y_0^2 + 2y_0)$. Then $y_0 = 0$ or $y_0 = -1$, which contradicts the fact that $y_0 > 0$. Thus $y_n \to \infty$. This completes the proof. $\square$

Put

$$U_1 = \{-1 < x < 1,\ y < 2(x - 1)\}, \quad L_1 = \{0 \leq x \leq 1,\ 2(x - 1) \leq y \leq 0\}.$$

Then $f(U_1) \subset U_0$, hence $U_1 \subset \mathbb{R}^2 \setminus K^{re}$.

Lemma 2.2. *$f(L_1) = \{0 \leq x \leq 1,\ -x + 2\sqrt{x} - 1 \leq y \leq 0\} \subset L_1$, hence $L_1 \subset K^{re}$.*

proof. Let $C_1 = \{0 \leq x \leq 1, y = x - 1\}$ be the set of critical points of q_x in L_1. Then the image $f(L_1)$ of L_1 sits above the image $f(C_1)$ of C_1. If $y = x - 1$, the point $(x_1, y_1) = f(x, y)$ satisfies $x_1 = x^2$ and $y_1 = y^2 + 2(1-x)y = -(x-1)^2$. Thus the set $f(C_1)$ is expressed by $y = -x + 2\sqrt{x} - 1$. Then $f(L_1) = \{0 \leq x \leq 1, -x + 2\sqrt{x} - 1 \leq y \leq 0\}$, which is included in $\{0 \leq x \leq 1, x - 1 \leq y \leq 0\} \subset L_1$. This completes the proof. $\square$

The preimage of the line $y = 2(x-1)$ is the curve $y^2 + 2(1-x)y = 2(x^2 - 1)$. Thus the set

$$U_2 = \{-1 < x < 0,\ y^2 + 2(1 - x)y < 2(x^2 - 1)\}$$

satisfies $f(U_2) \subset U_1$, hence $U_2 \subset \mathbb{R}^2 \setminus K^{re}$. On the other hand, the set

$$L_2 = \{-1 \leq x \leq 0,\ 2(x - 1) \leq y \leq 0,\ y^2 + 2(1 - x)y \geq 2(x^2 - 1)\}$$

satisfies $f(L_2) \subset L_1$, hence $L_2 \subset K^{re}$. This completes the proof of Theorem 2.1. $\square$

3 Basin of the attracting fixed point $(0, -1)$

Note that f has three fixed points $(0,-1), (0,0)$ and $(1,0)$. The point $(0,-1)$ is superattracting while the others are saddle. In this section, we consider the real slice of the basin $\mathcal{B}$ of $(0,-1)$.

Theorem 3.1. *The slice $\mathcal{B} \cap \mathbb{R}^2$ is equal to the interior of K'.*

proof. Since $f(L_2) \subset L_1$, we only have to show that the interior of L_1 is included in $\mathcal{B}$. By Lemma 2.2, $f(L_1) = \{0 \le x \le 1,\ -x + 2\sqrt{x} - 1 \le y \le 0\}$. Put

$$L_0 = \{0 \le x \le 1/2,\ 2x - 1 \le y < 0\}.$$

Lemma 3.1. $L_0 \subset \mathcal{B}$.

proof. Take a point $(x,y) \in L_0$ and put $(x_n, y_n) = f^n(x,y)$. Then

$$\begin{aligned} y_1 - (2x_1 - 1) &= y^2 + 2(1-x)y - 2x^2 + 1 = (y + 1 - x)^2 + x(2 - 3x) \ge 0, \\ y_1 - y &= y(y + 1 - 2x) \le 0. \end{aligned}$$

Thus $f(L_0) \subset L_0$ and the sequence $\{y_n\}$ is monotonely decreasing, hence converges to a point y_0, which must be a fixed point of $q_0(y) = y^2 + 2y$. Since $y_0 < 0$, $y_0 = -1$. On the other hand, $x_n \to 0$. This completes the proof. □

Now, take a point (x,y) in $int\, L_1$. By Lemma 2.2, (x_k, y_k) satisfies $-x_k + 2\sqrt{x_k} - 1 \le y_k < 0$ for any $k \ge 1$. Note that $-x + 2\sqrt{x} - 1 > 2x - 1$ if and only if $x < 4/9$. Since there exists $k \ge 0$ such that $x_k < 1/4 < 9/4$, it follows that $y_k > 2x_k - 1$, hence $(x_k, y_k) \in L_0$. By Lemma 3.1, $(x,y) \in \mathcal{B}$. This completes the proof of Theorem 3.1. □

4 Topology of the fiber Julia sets

In this section, we consider the dynamics of f on $\mathbb{C}^2$. Let K denote the set of points $(z,w) \in \mathbb{C}^2$ whose orbits are bounded, $K_z = \{w \in \mathbb{C}; (z,w) \in K\}$ be the *fiber filled Julia set* and $J_z = \partial K_z$ be the *fiber Julia set*. Set $\mathbb{D}_r(w_0) = \{|w - w_0| < r\}$ and $\mathbb{D} = \mathbb{D}_1(0)$.

Proposition 4.1. *For $0 \le x \le 1$, the sets K_x and J_x are connected.*

proof. In case $x = 0, 1$, $K_0 = \overline{\mathbb{D}_1(-1)}$ and $K_1 = \overline{\mathbb{D}}$ are connected. By Theorem 3.1, for $z = x \in (0, 1)$, the critical point $x - 1$ of q_x is contained in $\mathcal{B}$. By Proposition 2.3 in Jonsson [J], we conclude that K_x and J_x are connected for $0 < x < 1$. $\square$

Proposition 4.2. *For $0 \leq x \leq 1$, J_x is a circle and K_x is a disk.*

proof. The cases $x = 0, 1$ are trivial. Since $f|_{z=0}$ is hyperbolic, the existence of the holomorphic motion of J_0 is assured in Roeder [R]. Thus there exists $\epsilon > 0$ such that J_z and K_z are homeomorphic to J_0 and K_0 respectively for $|z| < \epsilon$. We will show that, if J_{x^2} is a circle, so is J_x. By Theorem 3.1, the critical point $x - 1$ of q_x is in $int\, K_x$. Therefore, $q_x : J_x \to J_{x^2}$ is a covering map of degree two. Thus, if J_{x^2} is a circle, J_x cannot have a self-intersection point, hence J_x must be a circle. For any $0 < x < 1$, there exists k such that $x^{2^k} < \epsilon$, hence by induction it follows that J_x is a circle. Since $J_x = \partial K_x$, K_x is a disk for any $0 < x < 1$. This completes the proof. $\square$

By Lemma 2.1, $0 \in J_x$ for any $0 < x < 1$, while $0 \notin J_1 = \partial\mathbb{D}$. Thus the map $z \mapsto J_z$ is discontinuous at $z = 1$. We consider the limit of J_x as $x \to 1$.

Theorem 4.1. *As $x \to 1$, $J_x \to K_1$ in the Hausdorff topology.*

proof. Since the map $z \mapsto K_z$ is upper semicontinuous, $\limsup_{x\to 1} J_x \subset \limsup_{x\to 1} K_x \subset K_1$. We will show that $K_1 \subset \liminf_{x\to 1} J_x$.

Since K_x is a disk, we can define the *fiberwise Böttcher coordinate* $\varphi_x : \mathbb{C} \setminus K_x \to \mathbb{C} \setminus \overline{\mathbb{D}}$ and *external rays* $R_x(t) = \varphi_x^{-1}(\{re^{2\pi it}; r > 1\})$ with angle $t \in \mathbb{R}/\mathbb{Z}$. See Proposition 2.6 in [J]. By Lemma 2.1, $R_x(0)$ is the positive real axis, hence lands at 0 for any $x \in (0, 1)$. Since $q_x(R_x(t)) = R_{x^2}(2t)$, the landing point of $R_x(1/2)$ is a q_x-preimage of 0, which must be $2(x - 1)$. The landing points of $R_x(\pm 1/4)$ are respectively $x - 1 \pm \sqrt{(3x + 1)(1 - x)}i$, the q_x-preimages of the landing point $2(x^2 - 1)$ of $R_{x^2}(1/2)$. These landing points are close to the origin if x is close to 1. For fixed ϵ and n, there exists $x(\epsilon, n)$ such that the landing points of the rays $R_x(j/2^n), 0 \leq j < 2^n$ belong to $\mathbb{D}_\epsilon(0)$ for $x > x(\epsilon, n)$.

We only have to show that, for any $\epsilon > 0$, there exists $x = x_\epsilon$ such that, if $x > x_\epsilon$, $\mathbb{D}_\epsilon(w) \cap J_x \neq \emptyset$ holds for any $w \in K_1$. If x is close to 1, q_x is close to $q_1(w) = w^2$. Then, for fixed n, the rays $R_x(j/2^n)$ are close to the rays $R(j/2^n) = \{re^{2\pi ij/2^n}; r > 0\}$. Now, let $R'_x(t) = R_x(t) \cap \overline{\mathbb{D}_2(0)}$, $R'(t) = R(t) \cap \overline{\mathbb{D}_2(0)}$ and let d_H denote the Hausdorff distance. Then, for any $\epsilon > 0$, there exists n_ϵ such that $d_H(R'(j/2^n), R'((j+1)/2^n)) < \epsilon$ for any $0 \leq j < 2^n$ and $n \geq n_\epsilon$. Then, there exists $x_\epsilon \geq x(\epsilon, n_\epsilon)$ such that, if $x_\epsilon < x < 1$, $d_H(R'_x(j/2^{n_\epsilon}), R'(j/2^{n_\epsilon})) < \epsilon$ holds for any $0 \leq j < 2^{n_\epsilon}$. Thus, it follows that, for any $\epsilon > 0$, $d_H(R'_x(j/2^{n_\epsilon}), R'_x((j+1)/2^{n_\epsilon})) < 3\epsilon$ holds for any $0 \leq j < 2^{n_\epsilon}$ and $x_\epsilon < x < 1$.

By lower semicontinuity of the map $z \mapsto J_z$, any point on $J_1 = \partial\mathbb{D}$ is approximated by points on J_x as $x \to 1$. Thus, between any of the consecutive rays $R_x(j/2^n)$ and $R_x((j+1)/2^n)$, there exists a point $w_j \in J_x$ close to J_1. Since, by Proposition 4.2, J_x is a circle, w_j is connected to the origin by an arc in J_x, which must pass between $R'_x(j/2^n)$ and $R'_x((j+1)/2^n)$.

Now, take $w \in K_1$. If $|w| < \epsilon$, then $\mathbb{D}_\epsilon(w) \cap J_x$ contains the origin. Hence we may assume $|w| \geq \epsilon$. Put $n = n_\epsilon$. There exists j such that w lies between $R_x(j/2^n)$ and $R_x((j+1)/2^n)$ or sits on one of them. If $x > x_\epsilon$, the 3ϵ-neighborhood of $d_H(R'_x(j/2^n)$ contains both $R'_x((j \pm 1)/2^n))$. Then, $\mathbb{D}_{3\epsilon}(w)$ intersects both $R_x(j/2^n)$ and $R_x((j+1)/2^n)$, hence it also intersects J_x since J_x passes between them. This completes the proof. $\square$

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